

## 5.7 Justifying the Laws

### A Solidify Understanding Task



CC BY Rumble Press  
<https://flic.kr/p/iASeZ8>

The Pythagorean theorem makes a claim about the relationship between the areas of the three squares drawn on the sides of a right triangle: *the sum of the area of the squares on the two legs is equal to the area of the square on the hypotenuse*. We generally state this relationship algebraically as  $a^2 + b^2 = c^2$ , where it is understood that  $a$  and  $b$  represent the length of the two legs of the right triangle, and  $c$  represents the length of the hypotenuse.

What about non-right triangles? Is there a relationship between the areas of the squares drawn on the sides of a non-right triangle? (Note: The following proof is based on *The Illustrated Law of Cosines*, by Don McConnell <http://www.cut-the-knot.org/pythagoras/DonMcConnell.shtml>)

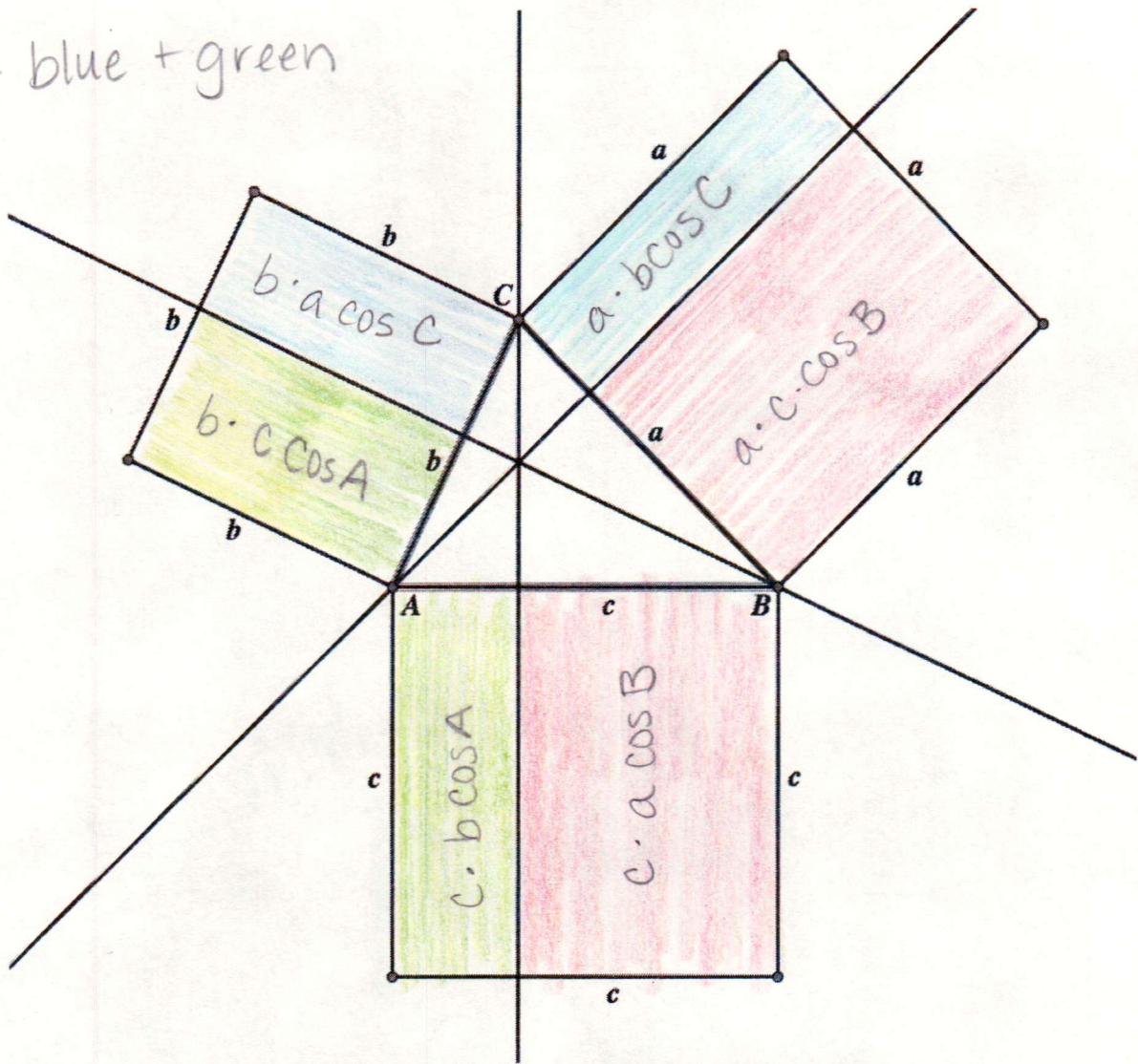
The diagram on the next page shows an acute triangle with squares drawn on each of the three sides. The three altitudes of the triangle have been drawn and extended through the squares on the sides of the triangle. The altitudes divide each square into two smaller rectangles.

1. Find an expression for the areas of each of the six small rectangles formed by the altitudes. Write these expressions inside each rectangle on the diagram. (Hint: The area of each rectangle can be expressed as the product of the side length of the square and the length of a segment that is a leg of a right triangle. You can use right triangle trigonometry to express the length of this segment.)
2. Although none of the six rectangles are congruent, there are three pairs of rectangles where each rectangle in the pair has the same area. Using three different colors—red, blue and green—shade pairs of rectangles that have the same area with the same color.
3. The area of each square is composed of two smaller, rectangular areas of two different colors. Write three different “equations” to represent the areas of each of the squares. For example, you might write  $a^2 = \text{blue} + \text{red}$  if those are the colors you chose for the areas of the rectangles formed in the square drawn on side  $a$ .



$a^2 = \text{blue} + \text{red}$

$b^2 = \text{blue} + \text{green}$



$c^2 = \text{red} + \text{green}$

4. Select one of your equations from step 3, such as  $a^2 = \text{blue} + \text{red}$ , and use the other two squares to substitute a different expression in for each color. For example, if in your diagram



$$4. \quad a^2 = \text{blue} + \text{red}$$

$$b^2 = \text{blue} + \text{green} \rightarrow \text{blue} = b^2 - \text{green}$$

$$c^2 = \text{red} + \text{green} \rightarrow \text{red} = c^2 - \text{green}$$

$$a^2 = b^2 - \text{green} + c^2 - \text{green}$$

$$a^2 = b^2 + c^2 - 2 \text{green}$$

$$\text{green} = b \cdot c \cdot \cos A$$

$$a^2 = b^2 + c^2 - 2(b \cdot c \cdot \cos A)$$

$$5. \quad a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$6. \quad b^2 = a^2 - \text{red} + c^2 - \text{red}$$

$$b^2 = a^2 + c^2 - 2 \text{red}$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$

$$c^2 = a^2 - \text{blue} + b^2 - \text{blue}$$

$$c^2 = a^2 + b^2 - 2 \text{blue}$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$



$blue = b^2 - green$  and  $red = c^2 - green$ , we can write this equation:  
 $a^2 = b^2 - green + c^2 - green$  or  $a^2 = b^2 + c^2 - 2 \cdot green$ .

Write your selected equation in its modified form here:

5. Since each color is actually a variable representing an area of a rectangle, replace the remaining color in your last equation with the expression that gives the area of the rectangles of that color.

Write your final equation here:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

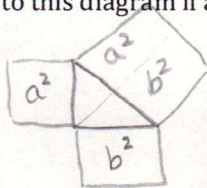
6. Repeat steps 4 and 5 for the other two equations you wrote in step 3. You should end up with three different versions of the **Law of Cosines**, each relating the area of one of the squares drawn on a side of the triangle to the areas of the squares on the other two sides.

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

7. What happens to this diagram if angle  $C$  is a right angle? (Hint: Think about the altitudes in a right triangle.)

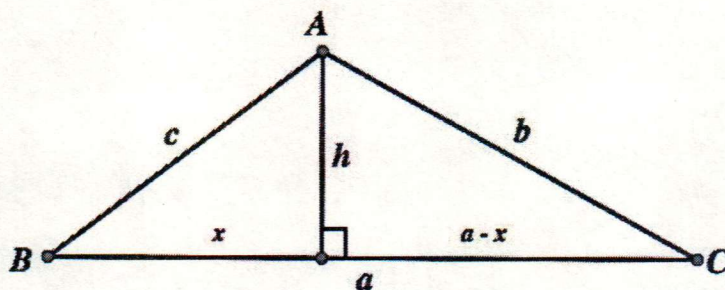


$$a^2 + b^2 = c^2$$

8. Why do we have to subtract some area from  $a^2 + b^2$  to get  $c^2$  when angle  $C$  is less than right?

The squares have more area equivalent to one of the three colors.

The Law of Cosines can also be derived for an obtuse triangle by using the altitude of the triangle drawn from the vertex of the obtuse angle, as in the following diagram, where we assume that angle  $A$  is obtuse.



9. Use this diagram to derive one of the forms of the Law of Cosines you wrote above. (Hint: As in the previous task, *More Than Right*, the length of the altitude can be represented in two different ways, both using the Pythagorean theorem and the portions of side  $a$  that form the legs of two different right triangles.)

see next page

10. Use the same diagram above to derive the **Law of Sines**. (Hint: How can you represent the length of the altitude in two different ways using sides  $a$ ,  $b$ , or  $c$  and right triangle trigonometry instead of the Pythagorean theorem?)

$$\sin B = \frac{h}{c} \rightarrow h = c \sin B$$

$$\sin C = \frac{h}{b} \rightarrow h = b \sin C$$

$$c \sin B = b \sin C \rightarrow$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$



$$9. c^2 = x^2 + h^2 \rightarrow c^2 - x^2 = h^2$$

$$b^2 = (a-x)^2 + h^2 \rightarrow b^2 - (a-x)^2 = h^2$$

$$b^2 - (a^2 - 2ax + x^2) = h^2$$

$$b^2 - a^2 + 2ax - x^2 = h^2$$

$$b^2 - a^2 + 2ax - x^2 = c^2 - x^2$$

$$b^2 - a^2 + 2ax = c^2$$

$$\cos B = \frac{x}{c}$$

$$x = c \cdot \cos B$$

$$b^2 - a^2 + 2a(c \cdot \cos B) = c^2$$

$$b^2 = c^2 + a^2 - 2a(c \cdot \cos B)$$

$$b^2 = c^2 + a^2 - 2ac \cdot \cos B$$